MATHEMATICS METHODS
MATHEMATICS YEAR 12, UNIT 3
TASK 2: TEST 1
Differentiation and Antidifferentiation

MELVILLE
SENIOR HIGH SCHOOL

Weighting: 8%

SECTION A: Calculator Free Section

TIME: 28 min

MARKS: 36

Student Name: SOLUTIONS

TO BE PROVIDED BY THE STUDENT
Standard Items: Pens, pencils, eraser, ruler.

INSTRUCTIONS TO STUDENTS:
You are required to attempt ALL questions,
Write answers in the spaces provided beneath each question.
Marks are shown with the questions.

Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be answered for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks.

It is recommended that students **do not use a pencil,** except in diagrams.

1. [2,2,3,3 = 10 marks]

Differentiate each of the following with respect to x. <u>Do not simplify</u>.

a. $y = 5x^2 + \sqrt[5]{x^2}$ [2]

not necessary

$$\frac{dy}{dx} = 10x + \frac{2}{5}x^{-\frac{3}{5}} \quad \left(= 10x + \frac{2}{5\sqrt[5]{x^3}}\right)$$

✓ ✓

b. $y = -\frac{7}{x^3} + 6\pi$ [2]

$$\frac{dy}{dx} = \frac{21}{x^4}$$ ✓✓

c. $y = (2x^5 - 1)(9 - 3x)^4$ [3]

$$\frac{dy}{dx} = 10x^4(9-3x)^4 + (2x^5-1)(4)(9-3x)^3(-3)$$

✓ ✓✓

d. $y = \frac{6x^3 + \sqrt{2}}{4x - 1}$ [3]

$$\frac{dy}{dx} = \frac{18x^2(4x-1) - 4(6x^3+\sqrt{2})}{(4x-1)^2}$$ ✓✓✓ (−1 per error)

2. [4 marks]

If $y = \frac{4}{h^2}$ and $h = x^5 + x$, demonstrate the use of the chain rule to determine $\frac{dy}{dx}$.

$$\frac{dy}{dh} = -\frac{8}{h^3}, \quad \frac{dh}{dx} = 5x^4 + 1$$ ✓

$$\frac{dy}{dx} = \frac{dy}{dh} \times \frac{dh}{dx}$$ ✓

$$= -\frac{8}{h^3} \times (5x^4+1)$$ ✓

$$= \frac{-8(5x^4+1)}{(x^5+x)^3}$$ ✓

3. [3, 3, 3 = 9 marks]

(a) Determine $\int (x-3)(x+1)dx$

(-1 overall if "+c" is missed)

$= \int (x^2 - 2x - 3)dx$ ✓

$= \frac{x^3}{3} - x^2 - 3x + c$ ✓✓

(b) Determine $\int \frac{2}{\sqrt{3x+1}} dx$

$= \int 2(3x+1)^{-\frac{1}{2}} dx$

$= \frac{2}{3} \int 3(3x+1)^{-\frac{1}{2}} dx$

$= \frac{2}{3} \frac{(3x+1)^{\frac{1}{2}}}{\frac{1}{2}} + c$ ✓✓

$= \frac{4}{3}\sqrt{3x+1} + c$ ✓

(c) Determine $\int 6x^2(1-x^3)^5 dx$

$= -2\int -3x^2(1-x^3)^5 dx$ ✓

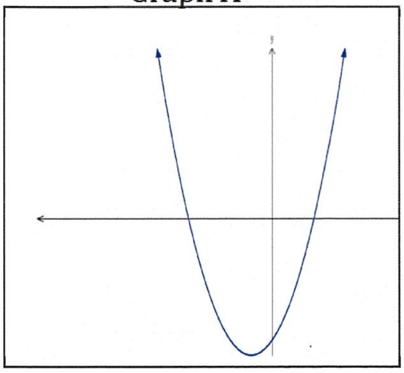
$= -\frac{2(1-x^3)^6}{6} + c$ ✓

$= -\frac{1}{3}(1-x^3)^6 + c$ ✓

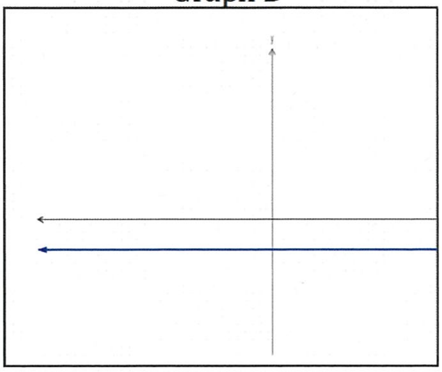
4. [3 marks]

Below are six graphs. 3 of the graphs are the derivative graphs of another 3 graphs shown. Match the three function graphs to their derivative graphs. ***Note: Some graphs may be used in more than one pair and other graphs may not be used at all.*** There is room at the bottom to place your answers.

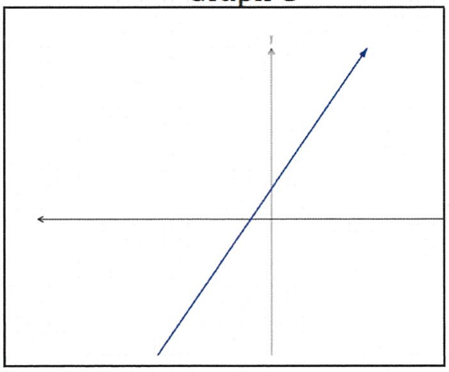
Graph A



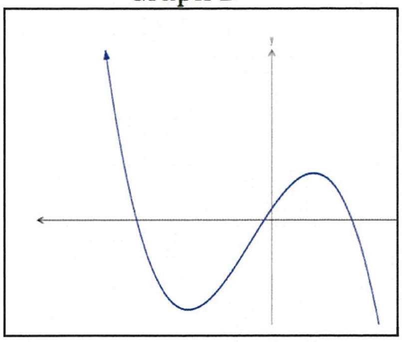
Graph B



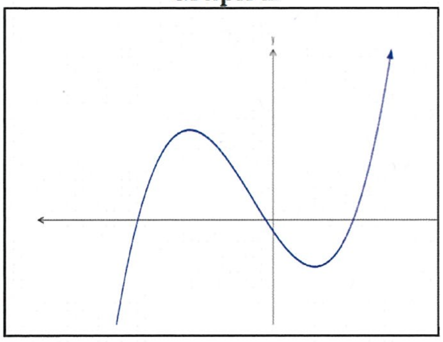
Graph C



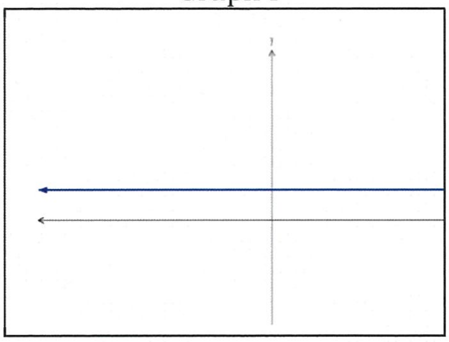
Graph D



Graph E



Graph F



Pair 1

Function: A

Derivative: C

Pair 2

Function: C

Derivative: F

Pair 3

Function: E

Derivative: A

5. [6 marks]

The equation of the tangent to the curve $y = ax^3 - bx^2 + 2$ when $x = -1$ is $y = 18x + c$.
The curve has a point of inflection when $x = 1$.
Find the values of a, b and c.

$y = ax^3 - bx^2 + 2$

$\frac{dy}{dx} = 3ax^2 - 2bx$

$\frac{dy}{dx}\Big|_{x=-1} = 3a + 2b = 18$ ① ✓

$\frac{d^2y}{dx^2} = 6ax - 2b$

$\frac{d^2y}{dx^2}\Big|_{x=1} = 6a - 2b = 0$ ✓

$3a = b$ ②

Substitute for b in ①

$3a + 2(3a) = 18$

$9a = 18$

$a = 2$ ✓

② ⇒ $b = 6$ ✓

So, $y = 2x^3 - 6x^2 + 2$

When $x = -1$,

$y = -6$ $(-1, -6)$ ✓

Now, substituting into $y = 18x + c$

$-6 = 18(-1) + c$

$12 = c$ ✓

$\therefore a = 2, b = 6, c = 12$

6. [4 marks]

Find y in terms of x given that $\frac{dy}{dx} = \frac{x^3 - 4}{x^2}$ and $y = 5$ when $x = 1$.

$\int \frac{x^3 - 4}{x^2}\,dx$

$= \int (x - 4x^{-2})\,dx$ ✓

$= \frac{x^2}{2} + \frac{4}{x} + c$ ✓

$(1,5) \Rightarrow$ $\frac{1}{2} + 4 + c = 5$

$c = \frac{1}{2}$ ✓

$\therefore y = \frac{1}{2}x^2 + \frac{4}{x} + \frac{1}{2}$ ✓

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Weighting: 8%

SECTION B: Calculator Assumed Section

TIME: 27 min

MARKS: 30

Student Name: SOLUTIONS

TO BE PROVIDED BY THE STUDENT
Standard Items: Pens, pencils, eraser, sharpener, correction tape/fluid, highlighters, ruler.
Special Items: Drawing instruments, templates.
A maximum of three CAS calculators satisfying the conditions set by the Curriculum Council.
A maximum of one unfolded A4 sheet (both sides) of notes may be taken into the test.

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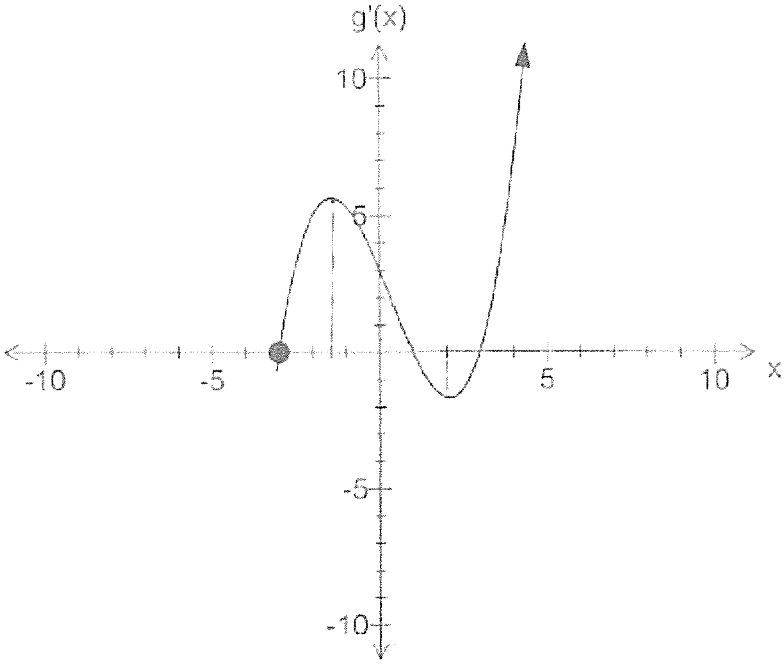
7. [1, 1, 2, 1, 2 = 7 marks]

The graph of $g'(x)$ is given below.



a) What can be said about the gradient of the function $g(x)$ between x = -3 to x = 1?

The gradient is positive ✓

b) When does the function, $g(x)$, have a negative gradient?

From $x = 1$ to $x = 3$ or $1 < x < 3$ ✓

c) State an equation for the tangent to the graph of $g(x)$ at x = 3.

$y = 0$ (or $y = c$, any constant) ✓✓

d) Find the value of x at which $g(x)$ has a relative maximum for $-3 \le x \le 4$

$x = 1$ ✓

e) Find the x-coordinate of each point of inflection of the graph of $g(x)$ for $-3 \le x \le 4$

$x = -1.5$ and $x = 2$
✓ ✓

8. [1,2,2,2 = 7 marks]

A particle is moving in rectilinear motion with acceleration 'a' at any time 't', in ms^{-2}, given as

$$a = 6t - 1$$

Initially, the particle is at the origin with a velocity of $-2m/s$.

Determine

a) The velocity of the particle at any time t.

$v = \int (6t-1)\,dt$

$= 3t^2 - t + c$ When $t=0, v=-2m/s$

$\therefore v(t) = 3t^2 - t - 2$ m/s ✓

b) When the particle is again at the origin.

$x = \int (3t^2 - t - 2)\,dt$

$= t^3 - \frac{t^2}{2} - 2t + C$ When $t=0, x=0$

$\therefore x(t) = t^3 - \frac{1}{2}t^2 - 2t$ ✓

$= 0$ when $t = 0, -1.186, 1.686s$

✓ $\therefore$ Particle is again at the origin when $t = 1.686s$ (or $\frac{\sqrt{33}}{4} + \frac{1}{4}$)

c) The minimum velocity of the particle.

$\frac{dv}{dt} = a = 6t - 1 = 0$

when $t = \frac{1}{6}s$ ✓

$\frac{d^2v}{dt^2} = 6 > 0$ for all values of t. $\therefore v(\frac{1}{6})$ is a minimum value

$v(\frac{1}{6}) = -2\frac{1}{12}$ m/s

or $= -2.08\dot{3}$ m/s ✓

d) The total distance travelled by the particle in the first three seconds.

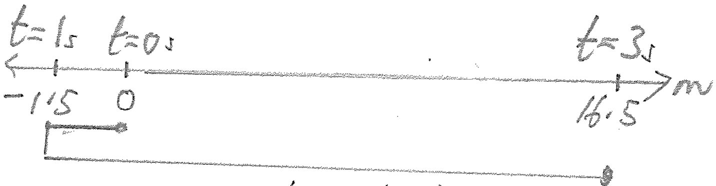
$\int_0^3 |3t^2 - t - 2|\,dt = 19.5$ m ✓✓

or $v(t) = 0$ when $t = 1$

So, $x(0) = 0$

$x(1) = -1.5$

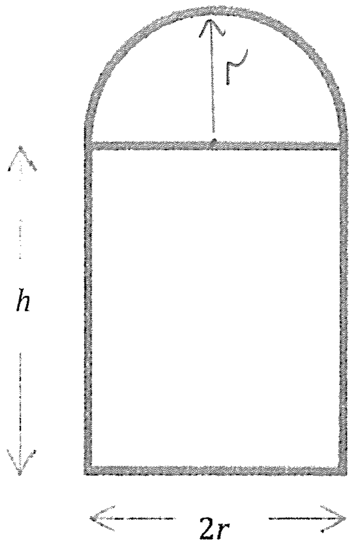
$x(3) = 16.5$



$\therefore$ 19.5m travelled.

9. [2, 3, 3, 3, 1 = 12 marks]

The diagram shows an arched church wooden window frame, to be made from 10m of timber.



a) Find an expression for h in terms of r.

Perimeter $= 2h + 4r + \pi r = 10$ ✓

$$h = \frac{10 - 4r - \pi r}{2}$$ ✓

b) Show that the area of the window is $A = 10r - r^2\left(4 + \frac{\pi}{2}\right)$

Area, $A = 2rh + \frac{1}{2}\pi r^2$

$$= 2r\left(\frac{10 - 4r - \pi r}{2}\right) + \frac{1}{2}\pi r^2$$ ✓

$$= 10r - 4r^2 - \pi r^2 + \frac{1}{2}\pi r^2$$ ✓

$$= 10r - r^2\left(4 + \frac{\pi}{2}\right)$$ ✓

Hence, or otherwise,

c) Show that the **exact** value of r that maximises the area is $r = \frac{10}{8+\pi}$

Solve $\frac{d}{dr}\left(10r - r^2\left(4+\frac{\pi}{2}\right)\right) = 0$ ✓✓

$\Rightarrow r = \frac{10}{8+\pi}$ m ✓

$\left(\frac{dA}{dr} = 10 - 2r\left(4+\frac{\pi}{2}\right) \right.$

$= 10 - 8r - \pi r$

OR $\left. 10 - (8+\pi)r \right)$

d) Suppose the radius (r) is increased by 10cm. Find the approximate change, using calculus methods, in the height of the window if the 10m of timber restriction still applies.

$\delta h \approx \frac{dh}{dr} \times \delta r$

✓ $\left\{ \begin{array}{l} h = \frac{10-4r-\pi r}{2} \\ \frac{dh}{dr} = -2-\frac{\pi}{2}, \; \delta r = 0.1 \end{array} \right.$

$\approx \left(-2-\frac{\pi}{2}\right) \times 0.1$ ✓

≈ -0.357 m (to 3dp) ✓

∴ Approximate change is a decrease of 0.357m.

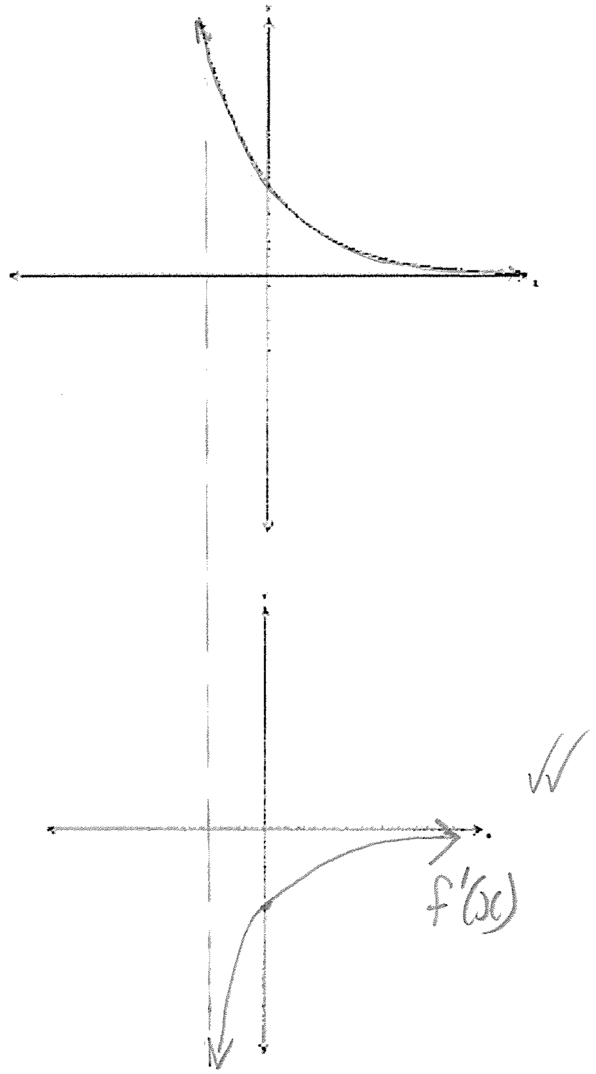
e) Interpret your answer in part (d).

An increase of 10cm in the radius results in a decrease of approximately 35.7cm in the height of the window. ✓

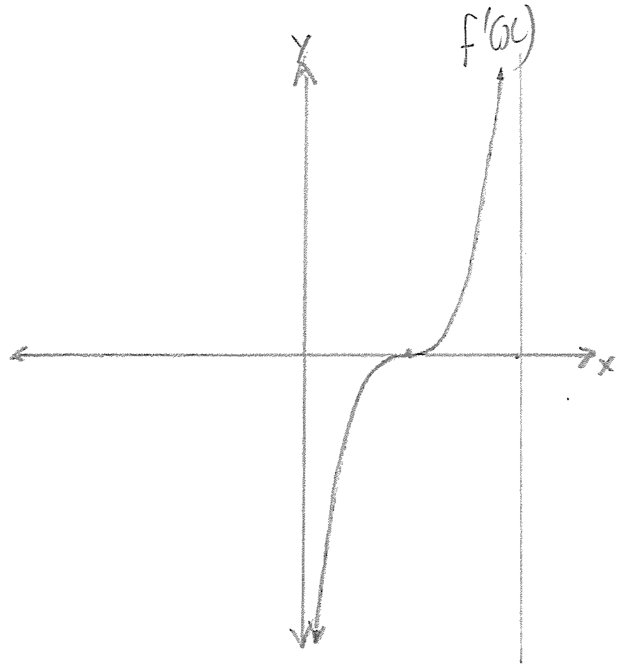
(Accepted comment of proportional change due to $\frac{dh}{dr}$ = a constant)

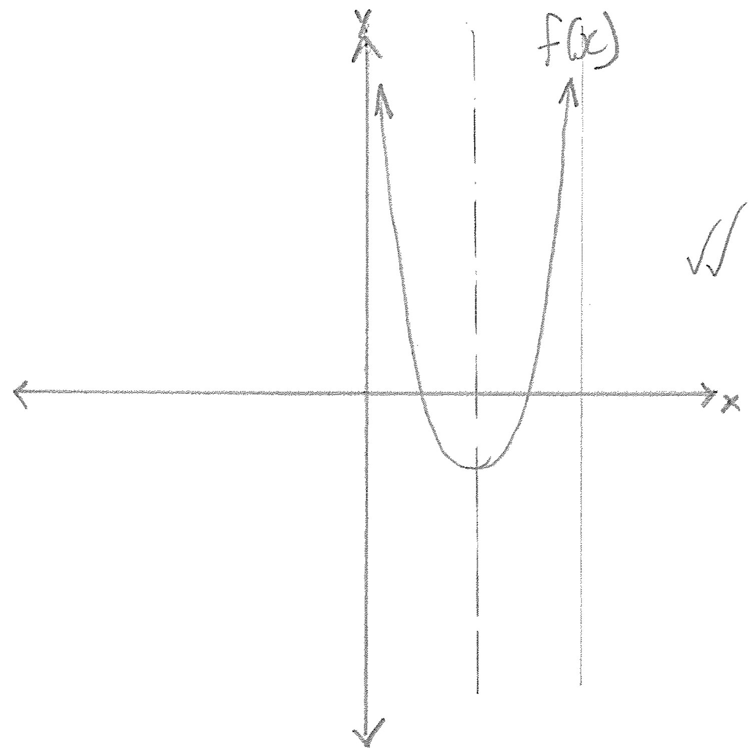
10. [2, 2 = 4 marks]

a) The graph of $f(x)$ is shown. Sketch the graph of the derivative function for $f(x)$ on the axes below.



b) <u>Given the derivative function</u>, sketch the graph of the function for on the axes below.





√√